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1996 J. Phys. A: Math. Gen. 29 6505

(http://iopscience.iop.org/0305-4470/29/20/008)

J. Phys. A: Math. Gen. **29** (1996) 6505–6513. Printed in the UK

The equation of motion for a spin vortex and geometric force

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Received 3 January 1996, in final form 20 May 1996

Abstract. The Hamiltonian equation of motion is studied for a vortex occurring in a two-dimensional Heisenberg ferromagnet of anisotropic type by starting with the effective action for the spin field formulated by the Bloch (or spin) coherent state. The resultant equation shows the existence of a geometric force that is analogous to the so-called Magnus force in superfluids. This specific force plays a significant role in the quantum dynamics of a single vortex, for example the determination of the bound state of the vortex trapped by a pinning force arising from the interaction of the vortex with an impurity.

1. Introduction

Quantum vortices in superfluids is one of most attractive subjects in condensed matter physics. Among many aspects in vortex phenomena, the dynamics of many vortices has been developed by starting with the relevant assumption on the boson superfluid [1], which reproduced the well known form for the Hamiltonian equation for the assembly of vortices [2]. The quantization based on the Hamiltonian equation has also been studied [3]. Recently, a refined formulation has been given for the quantum treatment for a superfluid vortex in the framework of the generalized Hamiltonian dynamics starting with the Landau–Ginzburg action [4]. Besides the superfluid vortex, other types of quantum vortex have been of interest for some time; typical is the vortex in the Heisenberg ferromagnet (see, for example, [5]). The occurrence of a vortex in a ferromagnet is quite natural, if one notes a close resemblance between the superfluid He^4 and the ferromagnet as a quantum condensate, especially in the vicinity of the ground state. Following the procedure developed for the superfluid, the Hamiltonian dynamics has been studied for vortices occurring in two-dimensional spin condensates [6].

Apart from the quantum dynamics of an assembly of vortices, there has been a long interest in the peculiar behaviour of the motion of a single vortex mainly inspired by the type II superconductors [7–9]. Among others, an interest is focused on the existence of a specific force called the Magnus force. This specific force is known to occur when a vortex moves in the uniform stream and plays a role in explaining some characteristic features of type II superconductors [7–9]. The Magnus force is also known to play a crucial role in some peculiar properties of the superfluid: the smallness of the critical velocity, the attenuation ratio of the second sound wave in the rotating superfluid He^4 and so on [10].

The purpose of this paper is to put forward the equation of motion for a spin vortex for the ferromagnetic system within the Hamiltonian formulation for a quantum vortex which

0305-4470/96/206505+09$19.50

has been previously developed [6]. As a consequence, we naturally arrive at a force of Magnus type. Indeed, if one considers the resemblance between the superfluid He and the ferromagnet as quantum condensates, it is natural to expect a realization of such an analogous force. This force should be called the 'geometric force', which differs from the ordinary force derived from a potential function. We also show another type of force called the pinning force, which comes from the interaction between a vortex and an impurity immersed in a condensate. It is shown that the effect of the pinning force is realized by the bound state of the vortex trapped in the pinning potential.

2. Spin field Lagrangian

Our starting point is the spin coherent state (or Bloch state) [11]. The quantum state for the spin system can be described by an infinite product of the Bloch state defined on each space point

$$|\{z(x)\}\rangle = \prod_{\boldsymbol{n}} |z_{\boldsymbol{n}}\rangle \tag{2.1}$$

where $\boldsymbol{n}$ means the vector assigning the lattice point, which is taken over continuous space points. Each component is given by the SU(2) coherent state

$$|z\rangle = \frac{1}{(1+|z|^2)^J} \exp[z\hat{J}_+]|0\rangle \tag{2.2}$$

where $|0\rangle = |J, -J\rangle$ is the lowest state satisfying $\hat{J}_-|0\rangle = 0$ and $\hat{J}_\pm$ are ladder operators and z takes any complex value. Using (2.1), we have the action function in the continuous limit

$$S = \int \langle\{z(x)\}|\mathrm{i}\hbar\frac{\partial}{\partial t} - \hat{H}|\{z(x)\}\rangle\, \mathrm{d}t = \int \mathcal{L}\, \mathrm{d}^2x\, \mathrm{d}t \tag{2.3}$$

where the Lagrangian density is given as

$$\mathcal{L} = \frac{\mathrm{i}J\hbar}{2}\frac{z^*\dot{z} - \dot{z}^*z}{1+|z|^2} - H(\{z(x)\}, \{z(x)\}^*). \tag{2.4}$$

Here note that the first term is a variant of the so-called 'geometric phase', which is represented in terms of the overcomplete set [12, 13]:

$$\Gamma = \oint \langle Z|\mathrm{i}\hbar\frac{\partial}{\partial t}|Z\rangle\, \mathrm{d}t. \tag{2.5}$$

If we are concerned with the quantization, it may be realized by constructing the propagator [11]

$$K = \langle z(x)| \exp\left[-\frac{\mathrm{i}H}{\hbar}T\right]|z(x)\rangle = \int \exp\left[\frac{\mathrm{i}}{\hbar}\int \mathcal{L}\, \mathrm{d}x\, \mathrm{d}t\right]\prod_{x,t} \mathrm{d}\mu[z(x,t)]. \tag{2.6}$$

The path integral may be used to derive the effective propagator for the vortex motion which leads to the semiclassical quantization of the vortex motion. By using the stereographic projection

$$z = \tan\frac{\theta}{2}\mathrm{e}^{-\mathrm{i}\phi} \tag{2.7}$$

with $(0 \leqslant \theta \leqslant \pi, 0 \leqslant \phi \leqslant 2\pi)$, the action is written as a familiar form

$$S = \int [\tfrac{1}{2}J\hbar(1-\cos\theta)\dot{\phi} - H(\theta,\phi)]\, \mathrm{d}^2x\, \mathrm{d}t. \tag{2.8}$$

Note that the first term is regarded as a field theoretic extension of the so-called ‘canonical term’, which has been well known in Hamiltonian dynamics for a spin

$$\int \frac{J\hbar}{2}(1-\cos\theta)\dot{\phi}\,\mathrm{d}t \equiv S_{\mathrm{C}}.$$

This term is also considered to be a generic Lagrangian in various quantum problems; for example the Lagrangian for the charged particle constrained on a sphere in the uniform magnetic field arising from the Dirac pole. The variation principle $\delta S = 0$ yields the field equation for the angle variables,

$$J\hbar\sin\theta\frac{\partial\theta}{\partial t} = -\frac{\delta H}{\delta\phi} \qquad J\hbar\sin\theta\frac{\partial\phi}{\partial t} = \frac{\delta H}{\delta\theta}. \tag{2.9}$$

3. Effective Lagrangian for a spin vortex

We shall derive the effective Lagrangian leading to the equation of motion for a single vortex. In order to achieve this, we need to construct the single vortex inherent in the static solution of the field equation for spin. In general this is given by an anisotropic Heisenberg model and the concrete form of this will be given in appendix A. We adopt the following feature: (i) the variable ϕ is given by an azimuthal angle

$$\phi = \tan^{-1}\frac{y}{x} \tag{3.1}$$

and (ii) the profile function θ is assumed to be given by a function of the radial variable r only, which is derived from the condition such that H takes an extremum. $\theta(r)$ may be solved by imposing the boundary condition for $\theta(r)$ such that the z-component spin $J_3(x) = J\cos\theta$ is directed upward inside the core (of radius a) and vanishes outside the core, namely, $\theta(r) \to 0$ for $r \to 0$ and $\theta(r) \to \pi/2$ for $r \to \infty$. This may be considered to be an idealization of the feature that the spin configuration is planar at infinity. $\theta(r)$, which incorporates this feature, may be simulated by the following form:

$$\theta(r) = \begin{cases} cr & (0 \leqslant r \leqslant a \equiv \pi/2c) \\ \pi/2 & (a \leqslant r). \end{cases} \tag{3.2}$$

Here the parameter c, which stands for the size of the vortex, should be determined so as to minimize the energy ((A5) in appendix A).

Having given the above preparation, we shall now turn to the dynamics for a single vortex. The dynamics can be built on the angle variables (θ, ϕ) such that the argument (x, y) is shifted by an amount of the coordinate for the vortex centre $(X(t), Y(t))$:

$$\phi(\boldsymbol{x}, t) = \mu\tan^{-1}\frac{y - Y(t)}{x - X(t)} \qquad \theta(\boldsymbol{x}, t) = \theta(\boldsymbol{x} - \boldsymbol{X}(t)) \tag{3.3}$$

where the coefficient μ stands for the vortex strength.

3.1. Canonical term

We first treat the canonical term which is written as

$$L_{\mathrm{C}} = \int \tfrac{1}{2}J\hbar(1-\cos\theta)\dot{\phi}\,\mathrm{d}^2x. \tag{3.4}$$

By making use of the chain rule,

$$\frac{\partial\phi}{\partial t} = \frac{\partial\phi}{\partial\boldsymbol{X}}\frac{\mathrm{d}\boldsymbol{X}}{\mathrm{d}t} \tag{3.5}$$

together with the fact that the gradient of the phase function gives the velocity field, namely, if we note the phase is given as a function of $\boldsymbol{x} - \boldsymbol{X}$, we get

$$\frac{\partial \phi}{\partial \boldsymbol{X}} = \nabla \phi = \boldsymbol{v} \tag{3.6}$$

where $\boldsymbol{v}$ denotes the velocity field coming from the vortex we are concerned with. Hence we get for the canonical term

$$L_{\mathrm{C}} = \int \tfrac{1}{2} J\hbar(1 - \cos\theta)\nabla\phi \cdot \dot{\boldsymbol{X}}\, \mathrm{d}x \tag{3.7}$$

which can also be written as the differential one form

$$\omega = \int \eta \boldsymbol{v} \cdot \mathrm{d}\boldsymbol{X}\, \mathrm{d}^2 x \tag{3.8}$$

where the following quantity is defined:

$$\eta = \tfrac{1}{2} J\hbar(1 - \cos\theta). \tag{3.9}$$

By using this notation, the boundary condition for θ assigned in the above can take over to the function η in the canonical term: $\eta \to 0$ (for $r \to 0$) and $\eta \to J\hbar/2$ (for $r \to \infty$). This feature is similar to the vortex for a bose fluid [4]. Furthermore, we note that the velocity field is given as

$$\boldsymbol{v} = \mu \boldsymbol{k} \times \nabla \log |\boldsymbol{x} - \boldsymbol{X}| \tag{3.10}$$

where $\boldsymbol{k}$ means the unit vector perpendicular to the (x, y) plane.

We shall now evaluate the effective Lagrangian for the motion of a vortex centre. By substituting (3.10) into (3.7), we get

$$\begin{aligned} L_{\mathrm{C}} &= \int \mathrm{d}^2 x\, \eta(\boldsymbol{x} - \boldsymbol{X})(\boldsymbol{v}(\boldsymbol{x} - \boldsymbol{X}) \cdot \dot{\boldsymbol{X}}) \\ &= I_1 \dot{X} + I_2 \dot{Y} \end{aligned} \tag{3.11}$$

where $I_\alpha (\alpha = 1, 2)$ are given by

$$I_\alpha = \int \eta v_\alpha\, \mathrm{d}^2 x. \tag{3.12}$$

The integrand of I_α does not decrease fast enough at large distance and the result depends on how one takes the limit, $r \to \infty$, on the integration boundary. This may be handled by a sort of 'regularization' and some care should be taken to get the correct results. Having carried out the regularization to evaluate the integral (see appendix B), the final result for the canonical term becomes

$$L_{\mathrm{C}} = \tfrac{1}{2} \eta_0 \mu (Y\dot{X} - X\dot{Y}). \tag{3.13}$$

If we note the boundary condition for $\theta = \pi/2$ at $r \to \infty$, the value of η_0 is given by

$$\eta_0 = \tfrac{1}{2} J\hbar. \tag{3.14}$$

Equation (3.13) is the central formula of the present theory.

3.2. Hamiltonian term

Next we consider the contribution that arises from the Hamiltonian. This is simply written in the form (see appendix A)

$$H = \int \mathcal{H}\, \mathrm{d}^2 x.$$

It can be easily verified that H does not depend on the vortex centre (X, Y). This feature comes from the following simple observation: the integrand is a function of $\boldsymbol{x} - \boldsymbol{X}$ and if we use the shift of the integration variable, $\boldsymbol{x} \to \boldsymbol{x} - \boldsymbol{X}$, the integral remains invariant, because even after making this change the integral should be carried over the whole two-dimensional plane. So the Hamiltonian simply gives a trivial contribution, which becomes nothing but the self-energy for the single vortex. It should be noted that this feature is peculiar to the case of the single vortex, namely, if we consider the general case for which there are several vortices, we should have the interaction term coming from the mutual configuration of the vortices.

3.3. Pinning potential by impurity

Finally we consider the role of the impurity effect for a vortex within the effective theoretical approach. It may be plausible that the interaction comes from the magnetic origin. The simplest choice satisfying this criterion may be given as follows

$$L_{\mathrm{pin}} = G \int \mathrm{d}^2 x\; J_3(x) s_3(x) \tag{3.15}$$

where G is a coupling constant. Here $s_3(x)$ represents the spin that is carried by a magnetic impurity. Let us consider the case that one magnetic impurity is located at $\boldsymbol{Y}$, and has the well localized density distribution. Then we can take the delta function approximation for such a spin disctribution

$$s_3(x) = s_3 \delta^2(\boldsymbol{x} - \boldsymbol{Y}) \tag{3.16}$$

where $s_3 = \pm\frac{1}{2}$. With the aid of the parametrization of the vortex form (3.2), the integral in (3.15) is calculated easily to result in the effective interaction between the vortex and impurity:

$$L_{\mathrm{impurity}}(\equiv) U_{\mathrm{pin}} = \begin{cases} G s_3 \cos[(\pi/2a)|\boldsymbol{X} - \boldsymbol{Y}|] & (|\boldsymbol{X} - \boldsymbol{Y}| < a) \\ 0 & (|\boldsymbol{X} - \boldsymbol{Y}| > a). \end{cases} \tag{3.17}$$

Here, if the coupling constant G is positive, the sign of spin s_3 is chosen such that the potential of the first half of (3.17) becomes attractive.

4. Geometric force and semiclassical bound state

4.1. Derivation of geometric force

Now combining (3.13) and (3.17), the effective Lagrangian for one vortex becomes

$$L_{\mathrm{eff}} = \frac{\eta_0 \mu}{2}(Y\dot{X} - X\dot{Y}) - H_{\mathrm{eff}}. \tag{4.1}$$

Here the second term is the Hamiltonian for the single vortex, which is nothing but the pinning potential; $H_{\mathrm{eff}} = U_{\mathrm{pin}}$. The equation of motion for the vortex centre can be obtained with the aid of the Euler–Lagrange equation

$$\delta \int L_{\mathrm{eff}}\, \mathrm{d}t = 0$$

which leads to

$$\dot{X} = \frac{\partial H_{\rm eff}}{\partial Y} \qquad \dot{Y} = -\frac{\partial H_{\rm eff}}{\partial X}. \tag{4.2}$$

The equation of motion (4.2) can be regarded as a special case of the canonical equation of motion [4, 6], where the pair (X, Y) should be regarded as a canonical variable. Now it is amazing to note that this canonical equation of motion is written as

$$\frac{\mu\eta_0}{2}\dot{\boldsymbol{X}} = \boldsymbol{k} \times \frac{\partial U_{\rm pin}}{\partial \boldsymbol{X}} \tag{4.3}$$

and, furthermore, this can be written by multiplying the vector $\boldsymbol{k}$, thus

$$\frac{\mu\eta_0}{2}(\boldsymbol{k} \times \dot{\boldsymbol{X}}) = -\frac{\partial U_{\rm pin}}{\partial \boldsymbol{X}} \tag{4.4}$$

where use is made of an elementary formula for the vector product

$$\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c}) = \boldsymbol{b}(\boldsymbol{a} \cdot \boldsymbol{c}) - \boldsymbol{c}(\boldsymbol{a} \cdot \boldsymbol{b})$$

together with the orthogonality relation:

$$(\boldsymbol{k} \cdot \nabla U_{\rm eff}) = 0.$$

From the physical point of view, the left-hand side of (4.4) is nothing but the 'geometric force', which is perpendicular to both the vortex velocity and the vector $\boldsymbol{k}$. This is analogous to the Magnus force in the case of the usual superfluid or the Lorentz force for the charged particle in a uniform magnetic field. On the other hand, the second term represents the pinning force that comes from the pinning potential, which is the usual force derived from the potential function. In this way, (4.4) can be regarded as a balance of the two kinds of forces.

4.2. Bound state by pinning potential

The motion of vortex can be quantized if we note that the vortex coordinate (X, Y) is a canonical variable. The quantization can be formally carried out by constructing the path integral

$$K_{\rm eff} = \int \exp\left[\frac{\rm i}{\hbar}\int \{\tfrac{1}{2}\eta_0\mu(Y\dot{X} - X\dot{Y}) - U_{\rm pin}\}\,{\rm d}t\right]\prod {\rm d}\mu[X(t), Y(t)]. \tag{4.5}$$

In order to get a crude estimate, the semiclassical limit is convenient. It is known that in the semiclassical limit the path integral leads to the Bohr–Sommerfeld (BS) quantization which is utilized to derive the bound-state spectra formed by the vortex trapped by the pinning potential. The BS rule is simply given by

$$\frac{\mu\eta_0}{2}\oint_C (X\,{\rm d}Y - Y\,{\rm d}X) = 2\pi n\hbar. \tag{4.6}$$

Here C means a loop which is defined as $U_{\rm pin} = E$. For simplicity, the centre of impurity is assumed to be placed at the origin $\boldsymbol{Y} = 0$, and $Gs_3 < 0$, so the energy contour is given by

$$Gs_3 \cos\frac{\pi}{2a}|\boldsymbol{X}| = E. \tag{4.7}$$

From this form, the loop C becomes a circle; $|\boldsymbol{X}| = \sqrt{X^2 + Y^2} \equiv \rho$, hence the quantization rule turns out to be

$$\frac{\mu\eta_0}{2}\pi\rho^2 = 2\pi n\hbar \tag{4.8}$$

with n an integer. Thus we get the energy spectrum

$$E_n = Gs_3 \cos\left[\frac{\pi}{a}\sqrt{\frac{n\hbar}{\mu\eta_0}}\right]. \tag{4.9}$$

The critical bound state is limited by $E_n = 0$, which means the inequality;

$$\frac{\pi}{a}\sqrt{\frac{n\hbar}{\mu\eta_0}} \leqslant \frac{\pi}{2}.$$

This leads to the condition for the quantum number n:

$$n\hbar \leqslant \tfrac{1}{4}\mu\eta_0 a^2. \tag{4.10}$$

5. Summary

We have studied a possible occurrence of the geometric force in a magnetic condensate. This force is analogous to the Magnus force in ordinary superfluids. The characteristic property is the nature of the 'transversality', namely, the force is perpendicular to the velocity of the 'particle (vortex)', which suggests that the force does not attribute to the energy dissipation. This feature is a characteristic of the Lorentz force, so the geometric force is a kind of Lorentz force. However, it should be noted that the analogy with the Magnus force is not complete, since in the magnetic condensate we have no supercurrent as in the case of superfluids.

As is seen from the above derivation, the geometric force is attributed to the canonical term or a variant of the geometric phase. From the point of view of formulation, the geometric force may be obtained as a special case of the dynamics of many vortices that has been previously given [6]. However, the effective Lagrangian for the single vortex can naturally incorporate the effect of the pinning force if we include the interaction with the magnetic impurities immersed in the magnetic substance. Indeed, we have shown that by using the Bohr–Sommerfeld quantization the geometric force results in the bound state of a vortex captured by a pinning potential. Apart from such a potential problem, the geometric force would play a role in the study of the effect of dynamical perturbation acting for the vortex motion. Details of this will be given elsewhere.

Note added in proof. Having submitted this paper, the authors were informed of the paper by Volovik (Volovik G 1986 *JETP Lett.* **44** 185) which suggests the existence of the Magnus force for a magnetic vertex. One of the authors (HK) would like to thank Professor Volvik at Helsinki University for his kind correspondence.

Appendix A

In this appendix we give a concrete form of the Hamiltonian which is not explicitly given in the text. We adopt the continuous version of the nearest-neighbour interaction of anisotropic type; the Hamitonian density is given by

$$\hat{\mathcal{H}} = \tfrac{1}{2}g\{(\nabla\hat{J}_1)^2 + (\nabla\hat{J}_2)^2 + \lambda(\nabla\hat{J}_3)^2\} \tag{A.1}$$

where the parameter λ means the degree of anisotropy, which is assumed to be $0 < \lambda \leqslant 1$, and this makes the system favour the planar spin configuration. The expectation value becomes

$$\mathcal{H} = \tfrac{1}{2}g\{(\nabla J_1)^2 + (\nabla J_2)^2 + \lambda(\nabla J_3)^2\}. \tag{A.2}$$

By using the angle variable representation of the spin field,

$$J_1 = J\sin\theta\cos\phi \qquad J_2 = J\sin\theta\sin\phi \qquad J_3 = J\cos\theta$$

$\mathcal{H}$ can be written as

$$\mathcal{H} = \tfrac{1}{2} g J^2 \{(\cos^2\theta + \lambda\sin^2\theta)(\nabla\theta)^2 + \sin^2\theta(\nabla\phi)^2\} \tag{A.3}$$

where the second term may be regarded as a 'fluid kinetic energy': for the spin fluid

$$T \equiv \tfrac{1}{2} g J^2 \int \sin^2\theta(\nabla\phi)^2 \,\mathrm{d}^2x. \tag{A.4}$$

Futhermore, if we consider that the lateral angle θ can be chosen as a function of the radial variable r only, we have the form for the Hamiltonian

$$H = \int \left[r(\cos^2\theta + \lambda\sin^2\theta)\left(\frac{\mathrm{d}\theta}{\mathrm{d}r}\right)^2 + \frac{1}{r}\sin^2\theta \right] \mathrm{d}r. \tag{A.5}$$

This Hamiltonian leads to the field equation for $\theta(r)$ via the variation principle, which would form the basis of the dynamics of the single vortex.

Appendix B

In the following we evaluate the integral which appears in the canonical term,

$$I_\alpha = \int \eta v_\alpha \,\mathrm{d}^2x. \tag{B.1}$$

We define two discs as

$$D_A = \{\boldsymbol{x};\, |\boldsymbol{x}| \leqslant R_A\} \qquad D_a = \{\boldsymbol{x};\, |\boldsymbol{x} - \boldsymbol{X}| \leqslant a\} \tag{B.2}$$

where a is the size of the vortex defined in (3.2) and R_A should be taken large enough for D_A to include D_a. Let us define $V(a, A)$ as the region

$$V(a, A) = D_A - D_a. \tag{B.3}$$

The integral I_α is indefinite at $r \to \infty$, so we must adopt a proper 'regularization' to get the finite result; the integral region is restricted to the finite region $V(a, A)$ (cut-off), and the infinite limit should be taken

$$I_\alpha = I_\alpha(a) + \lim_{R\to\infty} I_\alpha(R) \tag{B.4}$$

where each integral is written as

$$I_\alpha(a) = \int_{D_a} \mathrm{d}^2x\, \eta v_\alpha \qquad I_\alpha(R) = \int_{V(a;R)} \mathrm{d}^2x\, \eta v_\alpha. \tag{B.5}$$

By using the polar coordinates around $\boldsymbol{x} = \boldsymbol{X}$, the integral $I_\alpha(a)$ can be evaluated

$$\begin{aligned} I_\alpha(a) &= \int_{D_a} r\,\mathrm{d}r\,\mathrm{d}\theta \frac{\eta_0}{a} r \frac{\mu}{2\pi} \frac{\epsilon_{\alpha\beta} x_\beta}{r^2} \\ &= \frac{\eta_0\mu}{2\pi a}\epsilon_{\alpha\beta} \int_0^a r\,\mathrm{d}r \int_0^{2\pi} \hat{x}_\beta\,\mathrm{d}\theta = \frac{\eta_0\mu}{2\pi a}\epsilon_{\alpha\beta}\frac{a^2}{2}\cdot 0 = 0 \end{aligned} \tag{B.6}$$

where $\hat{x}_\alpha = x_\alpha / r$. With the aid of the Stokes theorem, the integral $I_\alpha(R)$ can be written as a line integral:

$$\begin{aligned} I_\alpha(R) &= -\epsilon_{\alpha\beta}\eta_0 \int_{V(a;R)} \mathrm{d}^2x \, \partial_\beta G_V(x;X) \\ &= -\epsilon_{\alpha\beta}\eta_0 \int_{\partial D_A + \partial D_a} \mathrm{d}s_\beta \, G_V(x;X). \end{aligned} \tag{B.7}$$

Here we define the function $G_V = \log|\boldsymbol{x} - \boldsymbol{X}|$. Furthermore, by using the polar coordinate around $x = X$, the integral on ∂D_a becomes

$$\int_{\partial D_a} \mathrm{d}s_\beta \, G_V(x;X) = -\frac{\mu}{2\pi} \log a \int \mathrm{d}s_\beta = 0 \tag{B.8}$$

whereas for around $\boldsymbol{x} = 0$, the integral on ∂D_A is evaluated to be

$$\begin{aligned} \int_{\partial D_A} \mathrm{d}s_\beta \, G_V(x;X) &= -\frac{\mu}{2\pi} \int \mathrm{d}s_\beta \left[\log R + \frac{(\boldsymbol{x}\boldsymbol{X})}{R^2} + \mathcal{O}(R^{-2}) \right] \\ &= -\frac{\mu}{\pi} \frac{1}{R^2} \int \mathrm{d}s_\beta (\boldsymbol{x}\boldsymbol{X}) + \mathcal{O}(R^{-2}) \\ &= \frac{\mu}{2\pi} X_\alpha \int \mathrm{d}\theta \, \hat{x}_\alpha \hat{x}_\beta + \mathcal{O}(R^{-2}) = \tfrac{1}{2}\mu X_\beta + \mathcal{O}(R^{-2}). \end{aligned} \tag{B.9}$$

Combining (B.8) and (B.9), we get the final result for I_α^V:

$$I_\alpha = \tfrac{1}{2}\eta_0 \mu \epsilon_{\alpha\beta} X_\beta. \tag{B.10}$$

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